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GENERALIZED EXTERNALITY GAMES: ECONOMIC APPLICATIONS.

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Abstract

This paper analyzes two specific economic problems: The division of profits between associated firms and the distribution of cost of a public good. We tackle these problems using game theoretic techniques. To do so, a new class of games in characteristic function form, called Generalizad Externality Games is defined in this paper. Some attractive features of this class of games are that the core is non-empty and that these new games there seems to be a connection with relational goods.

Key Words

Cooperative Games; Balanced Games; Core; Coob-Douglas Production; Public Goods; Relational Goods.

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1 Introduction

Cooperative Game Theory can be a useful tool in modelling situations in which economic agents cooperate. In many real life situations the problem of allocating joint profits and cost occurs. This paper develops an *analytical model-game* for analyzing how to divide the profits when some associated firms produce with a *Cobb-Douglas* technology, and how to divide the cost of a public good in proportion to benefits that agents derive from this good.

A new class of games in characteristic form, *Generalized Externality Games* are introduced in this paper. These new cooperative games have allocations belonging in the core, and therefore no coalition S will unanimously decide to challenge it since there is no way to divide $v(S)$ so as to make every member of S better off. Some of these allocations, for example *Shapley Value* and the *Proportional Solution*, solve problems such as sharing profits or costs, in some way, among the participants of a generalized externality game.

Morover, this new class of games, can be a useful tool in theoretical explanations of why rational individuals participate in groups with fees. The players do not wish simply to *be identified within* a group, but they also wish to be included, because the jointness of presence itself provides a benefit. For example, in a football game each agent pays a ticket, but the game is more exciting when the ground is full (one's *presence* benefits the rest of agents).

The paper is organized as follows. In Section 2, we introduce some concepts of cooperative game theory. In Section 3, we define a new class of games, *Generalized Externality Games*, and introduce some properties, showing that they are balanced games. In Section 4, we present two subclasses of *Generalized Externality Games.* We define the σ-generalized externality games and the constant-generalized externality games. We show that the σ-generalized externality games are average convex games. In Section 5, we analyze an allocation in the core of generalized games, the *Proportional Solution*, and we present an axiomatic characterization of this last solution. In Section 6, we study the following economic applications: *Cobb-Douglas* production Games and the *Provision of a One-Dimensional Public Good.* In this section performs, some simulations for particular values of the characteristic function are performed in order to investigate optimal distributions of total worth of a σ-generalized externality game. In Section 7, we consider the link between a model, which incorporates relational goods, and the improvement made by each player in a generalizaded externality game due to the fact that the presence of a player benefits the rest of the players. Conclusions are given in Section 8.

2 Preliminaries and basic definitions.

A cooperative game is a pair (N, v), where N is a finite set and v is a function from 2^N on $\Re$ called *characteristic function*, such $v(\emptyset) = 0$. The elements of $N = \{1, 2..\}$ are called players, the subset $S \in 2^N$, with $2^N = \{S \setminus S \subset N \ S \neq \emptyset\}$ are called *coalitions* and $v(S)$ is the *worth* of the coalition S. Let Γ_N be the set of all games (N, v), called *n-person games.*

In most interesting economic applications, the function v is superadditive or monotonic, so that it is efficient for the players to form the grand coalition, N.

A game $v \in \Gamma_N$ is called *superadditive* if:

$v(S) + v(T) \leq v(S \cup T)$, for all coalitions $S \cap T = \emptyset$.

A game $v \in \Gamma_N$ is called *monotonic* if:

$v(S) \leq v(T)$, for all coalitions $S \in T$.

Since the introduction of cooperative games, the problem most extensively studied in cooperative game theory is how to divide the total earning of the grand coalition if all players cooperate.

A *solution* on Γ_N is a function φ defined from Γ_N into $\Re^N$, such that $\sum_{i \in N} \varphi_i(v) = v(N)$ (this property is called efficiency).

Among the most popular multivalued concepts, the core proposes a very compelling solution. Formally, the core of the game is the set

$$C(v) = \{x \in R^N \ x(N) = v(N) \ and \ x(S) \geq v(S) \quad \forall \ S \subset N\},$$

where $x(S) = \sum_{i \in S} x_i$, for all S.

If an allocation belongs to the core, no coalition S will unanimously decide to challenge it since there is no way to divide $v(S)$ so as to make every member of S better off.

So thus, is very interesting to study the core of games cooperatives. We purpose in this paper games with non-empty core. We will make use of following definitions and results for to show that the core of those games is non-empty:

Definition: A *collection* $\mathcal{B}$ of coalitions is said to be *balanced* if there exist positive numbers Υ_S, for all $S \in \mathcal{B}$ (weights) such that, for each $i \in N$,

$$\sum_{\substack{S \in \mathcal{B} \\ i \in S}} \Upsilon_S = 1.$$

Definition: An n-person game v, with player set N, is *balanced* if, for every balanced collection $\mathcal{B}$, with balancing weights Υ_S,

$$\sum_{S\in\mathcal{B}} \Upsilon_S v(S) \leq v(N).$$

Theorem 1(Bondareva 1963, Shapley 1967): *A game (N, v) has a non-empty core iff it is balanced.*

We first provide sufficient conditions for the balancedness of a generalized externality game, which we will be define in the next section.

Other solution is the *Shapley value.* Shapley (1953), defined a value for games to be a function that assigns to each game v a number $\phi_i(v)$ for each i in N. He proposed that such a function obey three axioms (symetry axiom, carrier and additivity axiom). He showed that this unique value $\Phi = (\phi_i)_{i\in N}$ is

$$\phi_i(v) = \sum_{\substack{S\subset N \\ i\in S}} \frac{(s-1)!(n-s)!}{n!} \left[v(S) - v(S\backslash\{i\})\right],$$

with $s = card(S)$ and $n = card(N)$.

The core of an n-person game is the set of feasible outcomes that cannot be improved upon by any coalition of players. Shapley (1967), showed that the core of a *convex game* is non-empty and the Shapley value of a convex game is an element of the core.

Definition: A game $v \in \Gamma_N$ is *convex* if:

$$v(S \cup \{i\}) - v(S) \geq v(R \cup \{i\}) - v(R),$$

for all $S \subset N$ $R \subset S$ $\forall i \notin S$.

So for convex games the marginal contribution of each player to a large coalition is higher then his marginal contributation to a smaller coalition. But convexity is a strong requeriment and games that extend the convexity notion have been studied in the literature of cooperative games: For example, *Iñarra and Usategui* (1993), introduced the average convex games.

Definition: A game $v \in \Gamma_N$ is *average convex* if:

$$\sum_{i\in S} v(T) - v(T\backslash\{i\}) \geq \sum_{i\in S} v(S) - v(S\backslash\{i\}),$$

for all $S \subset T$.

They obtained for average convex games a interesting result:

Theorem 2(Iñarra and Usategui, 1993): *The Shapley value of an average convex games lies in the core.*

In this paper we study a class of games which is average convex and *Theorem 2* will be used in Section 4, and we compute some element of the core of that average convex games.

We want to tackle some economic problems with game theoretic techniques. To do so, we define a new class of game in characteristic function form. Really, we consider a generalized of a class of game called Externality games. *Grafe et al.* (1993), introduced the *Externality Games*, but they did not obtain economic interpretation from them.

Definition : A game $v \in \Gamma_N$ is said to be an *Externality Games* if there exists a vector $\beta = (\beta_i)_{i \in N} \in R_+^N \backslash \{0\}$ and non decreasing function $r : \{1, 2, ...n\} \longrightarrow R_+$ such that $v(S) = \beta(S)r(s)$ with $s = card(S)$.

We introduce a generalized of these externality games and we obtain a class of games with some economic applications.

3 Generalized Externality Games.

We define a new class of cooperative game and we study some properties. In the next sections, we obtain economic interpretations from these generalized of externality games. We call to this class of games *Generalized externality games.*

Definition: A game $v \in \Gamma_N$ is a *generalized externality game* if there exists a vector $\Pi \in R_+^N \backslash \{0\}$, $\Pi = (\pi)_{i \in N}$ and a real scalar $\alpha \geq 1$, and a non decreasing function, h, with $h : \{1, 2, ...n\} \longrightarrow R_+$, such that the payoffs are defined by:

$$v(S) = \Pi^\alpha(S)h(s),$$

where $\Pi^\alpha(S) = (\sum_{i \in S} \pi_i)^\alpha$, and s denotes the cardinal of S.

This class of games is denominated generalized externality games because the contribution of the player to a coalition is twofold. On the one hand, each player contributes with his particular endowment, π_i. On the other hand, his presence benefits the rest of the players (Grafe el al., 1993).

We denoted these games by $(\Pi^\alpha, h) \in GE_N$.

In the following propositions some properties of these games are examined.

First, we prove that it is efficient for the players to form the grand coalition, N.

Proposition 3.1 : *The games GE_N are superadditive and monotonic.*

Proof.- To show that a GE_N game is superadditive we use the following procedure: Let $(\Pi^\alpha, h) \in GE_N$ be a game, let S and T be coalitions T then

$$v(S) + v(T) = (\sum_{i \in S} \pi_i)^\alpha h(s) + (\sum_{i \in T} \pi_i)^\alpha h(t)$$

By the fact that h is non decreasing we obtain that

$$(\sum_{i \in S} \pi_i)^\alpha h(s) + (\sum_{i \in T} \pi_i)^\alpha h(t) \leq [(\sum_{i \in S} \pi_i)^\alpha + (\sum_{i \in T} \pi_i)^\alpha] h(s+t).$$

If we appied properties as $(a+b)^\alpha \geq a^\alpha + b^\alpha$ for all $\alpha \in \Re_+,\ \alpha \geq 1$, to $a = \Pi(S)$ and $b = \Pi(T)$ we obtain,

$$[(\sum_{i \in S} \pi_i)^\alpha + (\sum_{i \in T} \pi_i)^\alpha] h(s+t) \leq [\sum_{i \in S} \pi_i + \sum_{i \in T} \pi_i]^\alpha h(s+t) =$$

$$= [(\sum_{i \in S \cup T} \pi_i)^\alpha] h(s+t) = v(S \cup T). \ \square$$

The monotonic property is a direct consequence of the superadditive property applied to $T = \emptyset$. $\square$

Proposition 3.2: *The games GE_N are not convex.*

Proof.- Consider the following counter-example. Let $N = \{1,2,3\}, \Pi = (1,2,20), \alpha = 2$ and non decreasing function h given by $h(1) = 1,\ h(2) = 3,\ h(3) = 4.$ $\square$

Proposition 3.3: If we do not require any conditions on the function h, then the games GE_N are not average convex.

Proof.- Our purpose is to prove, with $\alpha = 1$, that given a game (Π^α, h) occurs the following condition (no average convex):

$$\sum_{i \in S} v(T) - v(T \backslash \{i\}) \leq \sum_{i \in S} v(S) - v(S \backslash \{i\}).$$

Equivalently,

$$\sum_{i \in S} [\sum_{j \in T} \pi_j h(t) + \sum_{j \in S \backslash \{i\}} \pi_j h(s-1)] \leq \sum_{i \in S} [\sum_{j \in S} \pi_j h(s) + \sum_{j \in T \backslash \{i\}} \pi_j h(t-1)]. \qquad [1]$$

If we defined the non decreasing function $\hat{h}$ following:

$$\hat{h}(t-1) = \hat{h}(t) = \hat{h}(s) = 1,$$

$$\hat{h}(s-1) < 1.$$

then we obtain,

$$\hat{h}(t)\alpha_1 + \hat{h}(s-1)\alpha_2 \leq \beta_1\hat{h}(s) + \beta_2\hat{h}(t-1),$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$.

Given that following equality always holds,

$\sum_{i\in T}\pi_j + \sum_{j\in S\setminus\{i\}}\pi_j = \sum_{j\in S}\pi_j + \sum_{j\in T\setminus\{i\}}\pi_j,$ for all $i \in S$.

We can conclude that the inequality [1] holds with the $\hat{h}$ defined in this proof.

If [1] holds then the games with characterist function $v(S) = (\sum_{i\in S}\pi_i)^\alpha \hat{h}(s)$ are not average convex. $\square$

Since the introduction of cooperative games, the most interesting problem studied is how to divide the total earning of the grand coalition if all players cooperate. Many different solution concepts have been proposed for n-person games in characteristic function form. We shall be concerned here with one of them: the core.

We want to prove that the core exists, so we can use the Bondareva and Shapley Theorem.

First we require,

Lemma 1: Sea h be a non-decreasing function, a real scalar $\alpha \geq 1$ and $\sum_{i\in S}\pi_i > \sum_{i\in T}\pi_i$ then

$$\frac{(\sum_{i\in W}\pi_i)^\alpha h(w)}{\sum_{i\in W}\pi_i} > \frac{(\sum_{i\in T}\pi_i)^\alpha h(t)}{\sum_{i\in T}\pi_i},$$

for all W, T such that $card(W) > card(T)$.

Proof.-

Let W, T be such that $card(W) > card(T)$, and $\sum_{i\in W}\pi_i > \sum_{i\in T}\pi_i$.

The function $f(x) = x^{\alpha-1}$ is increasing when $\alpha \geq 1$ then,

$$(\sum_{i\in W}\pi_i)^{\alpha-1} > (\sum_{i\in T}\pi_i)^{\alpha-1},$$

and so,

$$(\sum_{i\in W}\pi_i)^{\alpha-1}h(w) > (\sum_{i\in T}\pi_i)^{\alpha-1}h(t). \quad \square$$

Theorem 3.4 *Generalized externality games are balanced.*

Proof.- Let $\mathcal{B}$ be a balanced collection, with weights Υ_S.

$$\sum_{S\in\mathcal{B}} \Upsilon_S v(S) = \sum_{S\in\mathcal{B}} \Upsilon_S (\sum_{i\in S} \pi_i)^\alpha h(s) =$$

$$= \sum_{S\in\mathcal{B}} \Upsilon_S \left(\frac{(\sum_{i\in S} \pi_i)^\alpha h(s) \sum_{i\in S} \pi_i}{\sum_{i\in S} \pi_i} \right) =$$

$$= \sum_{i\in N} \pi_i \left(\frac{\sum_{\substack{S\in\mathcal{B}\\ i\in S}} \Upsilon_S (\sum_{i\in S} \pi_i)^\alpha h(s)}{\sum_{i\in S} \pi_i} \right) \leq$$

$$\leq \sum_{i\in N} \pi_i \sum_{\substack{S\in\mathcal{B}\\ i\in S}} \Upsilon_S \left(\frac{(\sum_{i\in N} \pi_i)^\alpha h(n)}{\sum_{i\in N} \pi_i} \right) = v(N),$$

where the last inequality follows from *Lemma 1* applied to $W = N$ and $T = S$ $\square$

Remark: If $\mathcal{B} = \{S_1, S_2, \ldots\}$ then

$$\sum_{S\in\mathcal{B}} \Upsilon_S \left(\frac{(\sum_{i\in S} \pi_i)^\alpha h(s) \sum_{i\in S} \pi_i}{\sum_{i\in S} \pi_i} \right) =$$

$$= \Upsilon_{S_1} \left(\frac{(\sum_{i\in S_1} \pi_i)^\alpha h(s_1) \sum_{i\in S_1} \pi_i}{\sum_{i\in S_1} \pi_i} \right) + \Upsilon_{S_2} \left(\frac{\sum_{i\in S_2} \pi_i)^\alpha h(s_2) \sum_{i\in S_2} \pi_i}{\sum_{i\in S_2} \pi_i} \right) + \ldots \quad [2]$$

Each player belong to any S_j $j = 1, 2 \ldots$, then

$$[2] \; = \pi_1 \sum_{\substack{S_j\in\mathcal{B}\\ 1\in S_j}} \Upsilon_{S_j} \left(\frac{(\sum_{i\in S_j} \pi_i)^\alpha h(s_j)}{\sum_{i\in S_j} \pi_i} \right) + \; \pi_2 \sum_{\substack{S_j\in\mathcal{B}\\ 2\in S_j}} \Upsilon_{S_j} \left(\frac{(\sum_{i\in S_j} \pi_i)^\alpha h(s_j)}{\sum_{i\in S_j} \pi_i} \right) + \ldots$$

$$= \sum_{i\in N} \pi_i \sum_{\substack{S\in\mathcal{B}\\ i\in S}} \Upsilon_S \left(\frac{(\sum_{i\in S} \pi_i)^\alpha h(s)}{\sum_{i\in S} \pi_i} \right).$$

Corollary 3.5: *The core of a generalized externality game is non-empty.*

Our first result shows that there do exist, at least, a feasible outcome that cannot be improved upon by any coalition of players. There is no way to divide $v(S)$ so as to make every member of S better off.

Totally balanced games were defined by Shapley and Shubik (1969) as those all of whose subgames have nooempty cores. They also proved that these games are the same as market games, which are generated by exchange economy the utility functions of whose traders are continuous and concave. Another characterization of totally balanced games provided by Kalai and Zemel (1982), who proved that they coincide with those games that can be expressed as the minimum of a finite collection of additive games. From this result they obteined that every totally balanced game is a flow game, i.e. a game associated with a direct network in such a way that the value of a coalition is defined as the maximum source to sink flow that can be sent by using only the edges which are owned by the members of the calition. Other caracterization of totally balanced games in terms of their associated indirect functions was introduced in Legaz (1992). This author studied a dual representation for n-person cooperative games, called the indirect function, which are characterized as certain nonincreasing polyhedral convex functions and many concepts in the theory of cooperative games can be easily expressed in terms of indirect functions. In the case of monotone games, the relationship between characteristic and indirect functions takes a simpler form.

Let us observe that Generalized Externality Games are balanced games and they admit a simple characterization in terms of indirect functions, since the core of (Π^{α}, r) can be represent in terms of its. We will study in a next paper a dual representation of Generalized Externality games based on indirect function.

4 Special cases of Generalized Externality Games.

In this Section we study two cases from generalized externality games, we do some restrictions on the function h. We will present, in Section 6, some economic illustration of these cases. We present the mathematical results that will be applied to the analysis of cooperative games in the following sections.

Let N be the set of players, a vector $\Pi \in \Re^N$ and a real scalar $\alpha \geq 1$.

We must consider two following case:

i) Let h be a non-decreasing function defined by

$$h(s) = s^{\sigma},$$

where $s = card(S)$ and $\sigma \in \Re_+$.

ii) Let h be a constant function defined by

$$h(s) = \mathcal{K},$$

where $s = card(S)$ and $\mathcal{K} \in \Re_+$.

These cases can define two classes of *generalized externality games*, which we develop in the next subsections.

4.1 σ-Generalized Externality Games

We study this case because the worth of a σ-generalized externality game, $v(S)$, can compute the maximum quantity of output that can be produced by any coalition S with a Cobb-Douglas technology.

Definition 4.1.1: A game $(\Pi^\alpha, h) \in GE_N$ is a *σ-generalized externality games* if h is non decreasing function, such that

$$h(s) = s^\sigma,$$

with $\sigma \in \Re_+$ and $s = card(S)$.

Remark: Now the characteristic function is $v(S) = (\sum_{i \in S} \pi_i)^\alpha s^\sigma$.

We prove that these games are a particular case of average convex games, which strictly include convex games, and then we can use the Theorem 2. It shows that the Shapley value is in the core of $\sigma - GE_N$.

Proposition 4.1.2 The σ-generalized externality games are average convex games.

Proof.- To show this proposition we follow the same procedure in I' narra and Usategui (1993).

Given $S \subset T$, and a game $(\Pi^\alpha, h) \in GE_N$.

We need to prove that

$$\sum_{i \in S} [\Pi^\alpha(S) s^\sigma - \Pi^\alpha(S \backslash \{i\})(s-1)^\sigma] \leq \sum_{i \in S} [\Pi^\alpha(T) t^\sigma - \Pi^\alpha(T \backslash \{i\})(t-1)^\sigma],$$

where $\Pi^\alpha(S) = (\sum_{i \in S} \pi_i)^\alpha$.

The last inequality is equivalent to

$$s s^\sigma \Pi^\alpha(S) - \sum_{i \in S} \Pi^\alpha(S \backslash \{i\})(s-1)^\sigma \leq s t^\sigma \Pi^\alpha(T) - \sum_{i \in S} [\Pi^\alpha(T \backslash \{i\})(t-1)^\sigma.$$

We prove first that

$$s t^\sigma \Pi^\alpha(T) - \sum_{i \in S} [\Pi^\alpha(T \backslash \{i\})(t-1)^\sigma \geq s t^\sigma \Pi^\alpha(S) - \sum_{i \in S} (t-1)^\sigma \Pi^\alpha(S \backslash \{i\}). \quad [3]$$

We have that $\Pi^\alpha(T) - \Pi^\alpha(T \backslash \{i\}) \geq \Pi^\alpha(S) - \Pi^\alpha(S \backslash \{i\})$.

Then

$$\sum_{i\in S}\Pi^\alpha(T)-\Pi^\alpha(T\backslash\{i\})\geq\sum_{i\in S}\Pi^\alpha(S)-\Pi^\alpha(S\backslash\{i\}).$$

The last inequality is equivalent to

$$s\Pi^\alpha(T)-s\Pi^\alpha(S)\geq\sum_{i\in S}\Pi^\alpha(T\backslash\{i\})-\Pi^\alpha(S\backslash\{i\}).$$

Then, we obtain the inequality [3],

$$t^\sigma[s\Pi^\alpha(T)-s\Pi^\alpha(S)]\geq t^\sigma[\sum_{i\in S}\Pi^\alpha(T\backslash\{i\})-\Pi^\alpha(S\backslash\{i\})]\geq(t-1)^\sigma[\sum_{i\in S}\Pi^\alpha(T\backslash\{i\})-\Pi^\alpha(S\backslash\{i\})].$$

To complete the proof we show that

$$st^\sigma\Pi^\alpha(S)-\sum_{i\in S}(t-1)^\sigma\Pi^\alpha(T\backslash\{i\})\geq ss^\sigma\Pi^\alpha(S)-\sum_{i\in S}\Pi^\alpha(S\backslash\{i\})(s-1)^\sigma. \quad [4]$$

Always,

$$s-1=\frac{\sum_{i\in S}\Pi(S\backslash\{i\})}{\Pi(S)}.$$

If $\alpha\geq 1$, we have

$$\sum_{i\in S}\frac{\Pi(S\backslash\{i\})}{\Pi(S)}\geq\sum_{i\in S}(\frac{\Pi(S\backslash\{i\})}{\Pi(S)})^\alpha.$$

Then, $(s-1)\Pi^\alpha(S)\geq\sum_{i\in S}\Pi^\alpha(S\backslash\{i\})$, and

$$(s-1)[(t-1)^\sigma-(s-1)^\sigma]\Pi^\alpha(S)\geq\sum_{i\in S}\Pi^\alpha(S\backslash\{i\})[(t-1)^\sigma-(s-1)^\sigma].$$

We must show that

$$s[t^\sigma-s^\sigma]\geq(s-1)[(t-1)^\sigma-(s-1)^\sigma].$$

If the last inequality occurs then we show [4] and the proof ends.

We consider the followig function , $f^\sigma(t,s)=s(t^\sigma-s^\sigma)$.

The inequality ,that we need to prove, holds if the last function defined satisfy this conditions:

$$f^\sigma(t,s)-f^\sigma(t-1,s-1)\geq 0.$$

If we applye the *Mean Value Theorem* [1]

[1] the * means scalar producte

$$f^\sigma(t,s)-f^\sigma(t-1,s-1) = (\nabla f^\sigma)_{(t',s')}*(t-(t-1),s-(s-1)) = \frac{\partial f^\sigma}{\partial t}(t',s')1+\frac{\partial f^\sigma}{\partial s}(t',s')1. \qquad [5]$$

We need to prove that the expression [5] is greater or equal than 0. This inequality holds if and only if: [2]

$$s\sigma t^{\sigma-1} + t^\sigma - s^\sigma + s\sigma s^{\sigma-1} \geq 0.$$

$$\Longleftrightarrow \frac{t^\sigma}{s^\sigma} \geq \frac{t(\sigma+1)}{s\sigma+t} \qquad \forall\sigma \in \Re_+.$$

If consider $\quad g(\sigma) = \frac{t^\sigma}{s^\sigma} - \frac{t(\sigma+1)}{s\sigma+t}$.

So $g(0) = 0 \quad$,

If we compute that $g(\sigma) \geq 0$ then we obtain that the inequality $[5] \geq 0$.

We apply the Taylors expansion, we have that:

$$g(\sigma) = g(0) + g'(0)(\sigma-0) + \frac{g''(\xi)}{2!}(\sigma-0)^2, \qquad 0 \leq \xi \leq \sigma.$$

We must prove that each member in the sum is non negative:

$$g'(\sigma) = (\frac{t}{s})^\sigma Ln(\frac{t}{s}) - \frac{t^2-ts}{(s\sigma+t)^2}.$$

In particular, $g'(0) = t^2 - \frac{t(t-s)}{Ln(t/s)} \geq 0$.

We have that,

$$g''(\sigma) = (Ln(t/s))^2(t/s)^\sigma + \frac{(t^2-ts)2s(s\sigma+t)}{(s\sigma+t)^4}.$$

In particular, $g''(\xi) \geq 0$

So as, $g(\sigma) \geq 0$ implies that the expression $[5] \geq 0$ holds, then we have shown that the generalized externality game, whose characteristic function is $v(S) = (\Pi(S))^\alpha s^\sigma, \quad \forall\sigma \in \Re_+ \quad \alpha \geq 1$, is an average convex games. $\square$

Proposition 4.1.2 allows to study some properties of the core of this $\sigma - GE_N$ games. A direct consequence of Shapley Theorem (1953) is this result:

Corollary 4.1.3: *The Value Shapley of a $\sigma - GE_N$ is an element of the core.*

[2] We denote (t',s') as (t,s) , now (t,s) is fixed.

4.2 Constant-Generalized Externality Games

In this subsection we study a class of generalizaded externality games, which are illustrated with some applications, for example, we study that the worth a constant-GE_N, $v(S)$, can compute the surplus generated by coalition S if they have to pay the full cost of a divisible public good.

Definition 4.2: A game $(\Pi^\alpha, h) \in GE_N$ is a *constant-generalized externality games* if the function h is such that

$$h(s) = \mathcal{K}$$

with $s = card(S)$, $\mathcal{K} \in \Re_+$.

Remark: Now the characteristic function is $v(S) = (\sum_{i \in S} \pi_i)^\alpha \mathcal{K}$, for all coalition S.

This class of games will be used in Section 6, there we will develop the relationship between this class of games and a general economic model where the agents share the cost of some abstract public decision.

We can compute the nucleolus for this case of games (with $\alpha = 2$). The nucleolus (Schemeidler, 1969) is the vector of payoffs that minimizes the maximal complaint of the coalitions in the following way. The excess (or complaint) of a coaliton $S \subset N$, $S \neq \emptyset$, with respect to a vector of payoffs $x \in \Re^N$ is defined by $e(S,x) = v(S) - \sum_{i \in S} x_i$. The excess vector of the payoff vector x is $e(x) = (e(S_1, x), \ldots, e(S_{2^N}, x))$, where the excesses are arranged in a decreasing order. The lexicographic order on $\Re^N$ is denoted by $\leq_L$. Hence, for $x, y \in \Re^N$ we have $x \leq_L y$ if and only if there exist a $k \in \{0, 1, \ldots, n\}$ such that $x_i = y_i$ for all $i \leq k$ and $x_{k+1} < y_{k+1}$. The nucleolus $\mathcal{V}(N, v)$ of a game (N, v) is the unique vector of payoffs for the game that minimizes $e(x)$ with respect to the order $\leq_L$.

As proved in Arín (1995), for games called $(\alpha, 2)$ the nucleolus is defined by $\mathcal{V}_i(\alpha, 2) = h(\alpha_i^2 + \alpha_i(\sum_{i \neq j} \alpha_j))$, where $v(S) = h[\alpha(S)]^2$ and h is a positive constant.

5 The Proportional Solution.

One of the main topics dealt with in cooperative game theory is given a game v, to divide the amonunt $v(N)$ between the players if the grand coalition N is formed. Many solutions concepts have been proposed to handle this problem. In this Section we study a element of the core, which is well-Known solution concept. First, we define a vector, call the proportional solution and then we prove that this solution belong to core of the generalized externality games.

Let N be a set of players. Consider a generalized externality game (Π^α, h) with a characteristic function defined by $v(S) = (\sum_{i \in S} \pi_i)^\alpha h(s)$.

Definition 5.1: A vector $x \in \Re_N$ is called *a proportional solution* for (Π^α, h) if each coordenate is defined by:

$x_i = \frac{\pi_i}{\sum_{i \in N} \pi_i} v(N)$, for all $i \in N$.

Proposition 5.2: *A proportional solution of a generalized externality game is an element of the core.*

Proof.-

First, we must prove that the vector x is well-defined. It is inmediate.

We develop the propieties of an element of the core:

i) $\sum_{i \in N} x_i = v(N)$

This equality is trivial.

ii) $\sum_{i \in S} x_i \geq v(S)$, for all $S \subset N$.

Now we have the following expression:

$$\sum_{i \in S} x_i = \sum_{i \in S} \pi_i \frac{v(N)}{\sum_{j \in N} \pi_j} = \sum_{i \in S} \pi_i \frac{(\sum_{i \in N} \pi_i)^\alpha h(n)}{\sum_{j \in N} \pi_j}. \qquad [6]$$

It is well known that the function $f(x) = x^{\alpha-1}$ with $\alpha \geq 1$ is non decreasing. Then

$$(\sum_{i \in N} \pi_i)^{\alpha-1} \geq (\sum_{i \in S} \pi_i)^{\alpha-1}.$$

Appliying the last inequality to expression [6] we obtain,

$$\sum_{i \in S} x_i \geq v(S) \quad for \quad all\ S \subset N. \qquad \square$$

6 Applications:

6.1 The provision of a One-Dimensional Public Good.

We introduce a general model (*Moulin*, 1992) where the agents share the cost of some abstract public decision. There is a divisible private good (money) used to produce public decisions.

We denote M an upper bound on anyone's endowment of money.

From now on the agents are assumed to produce a single divisible public good, $A = [0, +\infty]$. Denote by $c(a)$ the cost of producing decision $a \in A$. The cost function c is continuous, increasing in a such that $c(0) = 0,\ c(+\infty) = \infty$.

The set of agents jointly choosing a decision in A is $N = \{1, 2 \dots n\}$. A feasible outcome is a vector $(a; y_1, \dots, y_N)$ where:

$a \in A \quad y_i \leq M$ for all i and $\sum_{i \in N} y_i = c(a)$.

Agents $i's$ utility $u_i(a, y_i)$ is continuous, increasing in a and decreasing in y_i.

Consider the following case: *Quadratic Cost and Linear Utilities.*

We take $c(a) = a^2/2$ and $u_i(a, y_i) = \beta_i a - y_i$, where the parameter β_i is agent $i's$ marginal rate of substitution between private and public goods.

Consider the following *game 1*:

Let $N = \{1, 2, \dots, n\}$, the coalitions $S \subset N$, and the characteristic function $v(S)$ such that,

$$v(S) = max_{a \geq 0} \quad \sum_{i \in S} \beta_i\ a - \frac{a^2}{2}.$$

The worth of $v(S)$ compute the surplus generated by coalition S if they have to pay the full cost. If we consider the last *game 1* defined as a *constant-generalized externality game*[3], we have that the proporcional solution lies on the core.

Note that x_i, proportional solution, is:

$$x_i = \frac{\beta_i}{\sum_{i \in N} \beta_i} v(N) = \frac{\beta_i}{\sum_{i \in N} \beta_i} \frac{(\sum_{i \in N} \beta_i)^2}{2} = \beta_i \frac{\sum_{i \in N} \beta_i}{2}.$$

The Lindahl equilibrium solution is:

Total surplus $\sigma = (\beta_N)^2/2$; cost share $y_i = \beta_i \beta_N / 2$, with $\beta_N = \sum_{i \in N} \beta_i$.

With this result, it is easily seen that the Lindahl solution is in the core (the proportional solution is in the core). That solution divide the total cost of a public good between N agents just as the proportional solution divide the total worth of the surplus generated by N agents, who pay the full cost of a public good.

[3] The characteristic function is $v(S) = \Pi^{\alpha}(S) h(s)$ where $\Pi^{\alpha}(S) = \sum_{i \in S} (\beta_i)^2$ and $h(s) = 1/2$.

For a constant generalized externalility games given by $v(S) = \Pi^2(S)(1/2)$ the nucleolus coincides with the Shapley value. This result was obtained by Arín (1995), where both values can be easily expressed.

6.2 Cobb-Douglas Production.

Consider n firms $N = \{1, \ldots, n\}$ who can cooperate to produce a single joint product. The product is assumed to be perfectly divisible. For each subset of firms $S \subseteq N$, let $v(S)$ be the total amonunt produced by S when the agents in S joint their skills or recourses. We assume that nothing is produced for free; that is $v(\emptyset) = 0$. Every firm participates in production with *capital* and *labour*, which we denote by λ and η_i, respectively.

Thus, the resources owned by coalition S are:

$$\lambda(S) = s\lambda \quad \eta(S) = \sum_{i \in S} \eta_i.$$

We will assume that output q is obtained according to the following *Cobb-Douglas* technology with increasing returns to scale $(\beta \geq 1)$:

$$q = f(\lambda, \eta) = A\lambda^\alpha \eta^\beta.$$

The maximum quantity of output that can be produced by any coalition S will be

$$v(S) = f(\lambda(S), \eta(S)) = A\lambda^\alpha s^\alpha (\sum_{i \in S} \eta_i)^\beta. \quad [7]$$

The *Cobb-Douglas* production games [4] present a characteristic function given by the expression [7] and it is inmediately shows that these games are a particular case of a σ-*generalized externality games.*

If the production q is provided by a group of firms, then the profits that are generated by the output [5] have to be divided, in some way, among the participants. The goal of this subsection is to analyse this type of cooperative problems.

We can model some situation of cooperation between firms as a Cobb-Douglas Production Game, afterthat we use game theoretic solutions concepts studied for σ-generalized externality games. For example, we study two allocations belonging in the core: The *Shapley Value*

[4] Iñarra E., Usategui J. (1993) consider a nonlinear production game, namely a type of Cobb-Douglas Production Games with increasing returns to scale.

[5] The output can to be sold at a given market price.

and the *Proportional Solution.* To use the value Shapley or Proporcional Solution depends on the specific features of the firms and the properties that are required on the cooperation.

In many real life situations the problem of allocating joint profits occurs. In this Section, we solve an example where the production function is approximately a Cobb- Douglas. We present the results for simulations of the Cobb- Douglas game, where the parameters have been chosen that the technology must display increasing returns to scale.

Example and simulation:

Three retailers from Almeria have established a association call *Asociacion comerciante del poniente* in order to build a hypermarket. This is a proyect developed, managed and financed by retairles. They decided to meger in order to prevent a French multinational to get the conccession of this proyect. The invesment of 1200 millions of pesetas and 600 workers needed to put the hypermarket into operation is solely financed by 3 partners. The output (the value of hypermarket built with capital and labour) will divide between retailers, and we can use the Shapley value or proportional solution (both are optimal distributions).

We asume that $\lambda = 400$ millions of pesetas (the each firm's capital), $\eta_1 = 100$, $\eta_2 = 200$ and $\eta_3 = 300$ (the each firm's labour) and the parameters of a *Cobb-Douglas* are $\alpha = 1$ and $\beta = 1$.

TABLA 1

Coalitions	**(1)**	**(2)**	**(3)**	**(1,2)**	**(1,3)**	**(2,3)**	**(1,2,3)**
Payoffs	40000	80000	120000	240000	320000	400000	720000
Shapley Value	180000	240000	300000	–	–	–	–
Proportional solution	120000	240000	360000	–	–	–	–

As can be seen from the Table 1, the case considered is with increasing return to scale.

Observing Table 1, we have:

a) Any coalition act together will get no less than that when they act independently; Obviosly, $v(N)$ is then the largest amount of payoff that the player can possibly obtain.

b) The Shapley value and the proportional solution belong to core of this σ-generalized externality game.

Remark: We give two different way to distribute the joint maximum payoff $v(N)$ among all the n players.

7 Link between the relational goods and GE_N games.

Uhlaner (1989) develops a model, which incorporates a set of objetives, called *relational goods.*[6]

First, we make a brief summary about the relational goods, and next we study the connection between these goods and the generalized externality game.

People pursue *relational goods* which cannot be acquired by an isolated individual. Intead, these goods can only be *posessed* by mutual agreement that they exist after appropiate joint actions have been taken by a person and non-arbitrary others. For a relational good a person's utility increases both as his or her own consumption increases and as the consumption of some specific other person or member of a defined set of people increases. Relational goods are a subset of local public goods, however the jointness of consumption itself provides a benefit, and congestion can increase utility (A football game is more exciting when the ground is full).

The analysis of relational goods suggest circumstances under which participation is rational, for example, under some cicumstances persons will be more likely to act if they believe others will act, contrary to free-rider logic.

Uhlaner describe that relational goods can be categorized along two dimensions: By whether contact among those interacting is direct or indirect, and by whether the goods are instrumental or consumption goods.

In this Section, we explore the link between the instrumental relational goods and generalized externality games. Relational instrumental goods depend upon some policy outcome; For example, if action by one's group bolsters the group's political identity and individual participation is necessary as an *entry ticket* to claim group identity, there exists a relational instrumental good: the individual's share of augmented group identity. The feeling of being a real member of some group may be socially defined as requering certain actions. In this way, the individual does not get the benefit of the collective good without belonging to the group, and acting establishes a claim to membership.

We can think that the new class games defined in this paper (generalized externality game), ad also, suggest that individuals derive utility from the act of cooperation, so no only each player contributes with his endowment but his presence in order to increase the benefits to the rest of the players. A generalized externality game can reinforcement of sense of belonging to a group (my presence is important to the rest of the players, they are my group and they want to include me).

Uhlaner shows that the concept of relational goods fills in key gaps in the most promesing explanations of participation, we can say that the generalized externality games show that each player helps the cause or the group by being counted, his presence increase the total worth of game, it benefit the group (the rest of players) and then it can explain that the

[6]These goods depend upon interactions among persons.

people participate in some actions, belong to politics group (they know that his presence benefits to the rest members of group). Indead, they share of the augmented group identity if they participate in a generalized externality game.

8 Conclusions.

The preceding sections have defined a new class of cooperatives games, called generalized externality games. We prove that these games are balanced, that is, are games with a non-empty core. We study some allocations belongs to the core, such as proportional solution and the Shapley value. We obtain some applications of these games: The division of the surplus generated by a group if they have to pay the full cost of a divisible public good, and the distribution of the output when consider a group of firms who can cooperate to produce a single joint product with Cobb-Douglas technology. Finally, we express the idea that the players share of augmented group identity (each player helps by being counted) if they participate in a generalized externality game.

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